

1. (i) (a) Line L meets plane P when $2t + 3 + 3(4t - 1) - 2(2 - t) = 4$
 $\Rightarrow t = \frac{1}{2}$.

The required point of intersection is $(4, 1, 1\frac{1}{2})$. (M1) (A1)

- (b) $\vec{a} = 2\vec{i} + 4\vec{j} - \vec{k}$ is parallel to line L , and $\vec{n} = \vec{i} + 3\vec{j} - 2\vec{k}$ is normal to plane P .

If θ is the required angle, then

$$\begin{aligned}\sin\theta &= \frac{|\vec{a} \cdot \vec{n}|}{|\vec{a}| |\vec{n}|} \\ &= \frac{16}{\sqrt{21} \sqrt{14}} \\ &= \frac{16}{7\sqrt{6}}\end{aligned}$$

The required angle = 68.9° (nearest 0.1°). (M2) (A1)

- (c) $\vec{b} = -4\vec{i} + 2\vec{j} + \vec{k}$ is parallel to line M , and $\vec{b} \cdot \vec{n} = 0$.

Therefore, line M is parallel to plane P . (R1) (A1)

- (d) If line L meets line M , the equations

$$\begin{aligned}2t + 3 &= 6 - 4u \\ 4t - 1 &= 2u \\ 2 - t &= u - 10\end{aligned}$$

must have a unique solution.

From the first and second, $t = \frac{1}{2}$, $u = \frac{1}{2}$.

Now, these do not satisfy the third equation $\left(2 - \frac{1}{2} \neq \frac{1}{2} - 10\right)$.

Therefore, lines L and M do not meet. (M2) (A2)

(e) Method 1

Let $A(2t + 3, 4t - 1, 2 - t)$ lie on L and
 $B(6 - 4u, 2u, u - 10)$ lie on M .

$$\text{Now, } \vec{AB} = \begin{pmatrix} 3 - 4u - 2t \\ 1 + 2u - 4t \\ -12 + u + t \end{pmatrix}. \quad (M1) (A1)$$

$$\text{Also, the vector } \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & -1 \\ -4 & 2 & 1 \end{vmatrix} = 6\vec{i} + 2\vec{j} + 20\vec{k} \text{ is perpendicular to both lines } L \text{ and } M. \quad (A1)$$

If A and B are the points which are closest together, \vec{AB} would be parallel to $\vec{a} \times \vec{b}$. (R1)

$$\text{Therefore, } 3 - 4u - 2t = 3(1 + 2u - 4t) \Rightarrow 10t - 10u = 0$$

$$\text{and } -12 + u + t = 10(1 + 2u - 4t) \Rightarrow 41t - 19u = 22.$$

$$\text{Solving gives } u = t = 1, \text{ and so } \vec{AB} = \begin{pmatrix} -3 \\ -1 \\ -10 \end{pmatrix}.$$

$$\text{Therefore, the shortest distance} = \sqrt{110}. \quad (A1)$$

Method 2

$$\vec{a} \times \vec{b} = 6\vec{i} + 2\vec{j} + 20\vec{k} \text{ is perpendicular to both lines } L \text{ and } M. \quad (A1)$$

The plane $3x + y + 10z = 28$ contains the line L and is parallel to M . (R1)

Now, the required distance is the distance from the point $(6, 0, -10)$, which lies on M , to this plane.

$$\text{The required distance} = \frac{|18 + 0 - 100 - 28|}{\sqrt{110}} = \sqrt{110}, \quad (M2) (A1)$$

Method 3

$\vec{\omega} = 3\vec{i} + \vec{j} + 10\vec{k}$ is perpendicular to both lines. (AI)

$A(3, -1, 2)$ lies on L ; $B(6, 0, -10)$ lies on M . (AI)

The required distance is the orthogonal projection of \bar{AB} on $\vec{\omega}$. (RI)

$$\text{This is } \frac{|A\bar{B} \cdot \vec{\omega}|}{|\vec{\omega}|} = \frac{1}{\sqrt{110}} \left| \begin{pmatrix} 3 \\ 1 \\ -12 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 10 \end{pmatrix} \right| = \sqrt{110}. \quad (M1)(AI)$$

$$(ii) \quad (a) \quad I_0(x) = \int_0^x e^{-t} dt = -e^{-t} \Big|_0^x = 1 - e^{-x}. \quad (AI)$$

$$\begin{aligned} (b) \quad I_n(x) &= \int_0^x t^n e^{-t} dt \\ &= [-t^n e^{-t}]_0^x + n \int_0^x t^{n-1} e^{-t} dt \\ &= -x^n e^{-x} + n I_{n-1}(x), \text{ as required.} \end{aligned} \quad (M2) \quad (M1)AG$$

$$\begin{aligned} (c) \quad I_3(x) &= -x^3 e^{-x} + 3I_2(x) \\ &= -x^3 e^{-x} + 3(-x^2 e^{-x} + 2I_1(x)) \\ &= -x^3 e^{-x} - 3x^2 e^{-x} + 6(-x e^{-x} + I_0(x)) \\ &= -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} + 6 - 6e^{-x} \end{aligned}$$

Therefore, $I_3(1) = 6 - 16e^{-1}$. (M1) (AI)

2. (a) $f(x) = \frac{1 + \ln x}{x}$

At A, $\ln x + 1 = 0 \Rightarrow x = e^{-1}$.

Therefore, $A = (e^{-1}, 0)$.

(M1) (A1)

(b) $f'(x) = \frac{x\left(\frac{1}{x}\right) - (1 + \ln x)}{x^2} = \frac{-\ln x}{x^2},$ (A2)

and this is zero when $x = 1$.

Therefore, $B = (1, 1)$.

(M1) (A1)

(c) $f''(x) = \frac{x^2\left(-\frac{1}{x}\right) + 2x\ln x}{x^4} = \frac{2\ln x - 1}{x^3},$ (A2)

and this is zero when $\ln x = \frac{1}{2}$ or $x = e^{1/2}$. (A1)

Clearly, $f''(x)$ changes sign at $x = e^{1/2}$, and so $C = (e^{1/2}, \frac{3}{2}e^{-1/2})$. (M1) (A1)

(d) The gradient of $(OT) = f'(x_0) = -\frac{\ln x_0}{x_0^2}.$ (M1)

Therefore, the equation of (OT) is $y = \left(-\frac{\ln x_0}{x_0^2}\right)x.$ (A1)

Note: At T, $f'(x_0) = \frac{f(x_0)}{x_0}.$

$$\therefore \frac{-\ln x_0}{x_0^2} = \frac{1 + \ln x_0}{x_0^2}$$

Hence the equation of OT is $y = \left(\frac{1 + \ln x_0}{x_0^2}\right)x.$

Award (M1) (C1)

(e) Since T lies on both the curve and the tangent line,

$$\frac{1 + \ln x_0}{x_0} = \left(-\frac{\ln x_0}{x_0^2} \right) x_0. \quad (R1)$$

$$\Rightarrow \ln x_0 = -\frac{1}{2}. \quad (M1) (A1)$$

Therefore, $x_0 = e^{-1/2}$.

(f) The x -coordinates are: $x_A = e^{-1}$, $x_T = e^{-1/2}$, $x_B = 1$, $x_C = e^{1/2}$.

$$\text{Now, } \frac{x_T}{x_A} = \frac{x_B}{x_T} = \frac{x_C}{x_B} = e^{1/2}, \text{ which is constant.} \quad (M1)$$

Therefore, the x -coordinates are 4 consecutive terms of a GP, and
the common ratio is $e^{1/2}$.

(R1) (A1)

(g) The required area = Area of ΔODT - Area of ADT

$$\begin{aligned} &= \frac{1}{2} e^{-1/2} \left(\frac{1}{2} e^{1/2} \right) - \int_{e^{-1}}^{e^{-1/2}} \frac{1 + \ln x}{x} dx \\ &= \frac{1}{4} - \left[\frac{1}{2} (1 + \ln x)^2 \right]_{e^{-1}}^{e^{-1/2}} \quad (M1) (A1) \\ &= \frac{1}{4} - \frac{1}{2} \left(\frac{1}{4} - 0 \right) \\ &= \frac{1}{8} \quad (A1) \end{aligned}$$

3. (i) (a) In 6 experiments, $p(\text{exactly 2 successes}) = 3p(\text{exactly 3 successes})$

$$\Rightarrow \binom{6}{2} p^2(1-p)^4 = 3 \binom{6}{3} p^3(1-p)^3 \quad (M2)$$

$$\Rightarrow 15p^2(1-p)^4 = 60p^3(1-p)^3$$

$$\Rightarrow 1 - p = 4p$$

$$\Rightarrow p = \frac{1}{5} \quad (A2)$$

(b) $p(\text{at least one success in } n \text{ experiments}) > 0.99$

$$\Rightarrow 1 - p(\text{no success in } n \text{ experiments}) > 0.99 \quad (M1)$$

$$\Rightarrow 1 - \left(\frac{4}{5}\right)^n > 0.99 \quad (A1)$$

$$\Rightarrow \left(\frac{4}{5}\right)^n < 0.01 \quad (A1)$$

$$\Rightarrow 1.25^n > 100 \quad (A1)$$

$$\Rightarrow n > \frac{2}{\log 1.25} \approx 20.6 \quad (A1)$$

Therefore, the least number of times = 21. (A1)

(ii) (a) Mean = $np = 20 \times 0.06 = 12$.

$$\text{Standard deviation} = \sqrt{np(1-p)} = \sqrt{12 \times 0.94} = 3.36. \quad (A2)$$

$$(b) \text{ We require } p(X \geq 20) = p\left(Z > \frac{19.5 - 12}{\sqrt{11.28}}\right) \quad (A2)$$

$$= p(Z > 2.233) \quad (A1)$$

$$= 1 - p(Z < 2.233) \quad (A1)$$

$$= 1 - 0.9872$$

$$= 0.0128 \quad (A1)$$

4. (i) (a) $2z^2 - (2 - 2i)z - 5i = 0$

$$\begin{aligned} \text{Therefore, } z &= \frac{2 - 2i \pm \sqrt{(2 - 2i)^2 + 40i}}{4} \\ &= \frac{2 - 2i \pm \sqrt{32i}}{4} \\ &= \frac{1}{2}(1 - i) \pm \sqrt{2i}. \end{aligned} \quad (M1) (A2)$$

Without loss of generality, we may choose

$$z_1 = \frac{1}{2}(1 - i) + \sqrt{2i} \text{ and } z_2 = \frac{1}{2}(1 - i) - \sqrt{2i}. \quad AG$$

(b) Since $(2 - 2i)^2 = -8i$, [see part (a)], $(2 + 2i)^2 = 8i$ and so $(1 + i)^2 = 2i$. *(M1) (A1)*

$$\begin{aligned} \text{Therefore, } z_1 &= \frac{1}{2}(1 - i) + (1 + i) \text{ and} \\ z_2 &= \frac{1}{2}(1 - i) - (1 + i). \end{aligned}$$

$$\text{i.e. } z_1 = \frac{3}{2} + \frac{1}{2}i \text{ and } z_2 = -\frac{1}{2} - \frac{3}{2}i. \quad (M1) (A1)$$

[or vice versa since $\sqrt{2i} = \pm(1 + i)$]

(ii) (a) If $z = \cos\theta + i\sin\theta$, then $z^n = \cos n\theta + i\sin n\theta$, and $z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$. (De Moivre) *(M1) (A1)*

$$\text{Therefore, } z^n + \frac{1}{z^n} = 2\cos n\theta. \quad (M1) AG$$

$$(b) \quad 5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$$

$$\Rightarrow 5z^2 - 11z + 16 - \frac{11}{z} + \frac{5}{z^2} = 0 \text{ (since } z \neq 0\text{)} \quad (M1) (R1)$$

$$\Rightarrow 5\left(z^2 + \frac{1}{z^2}\right) - 11\left(z + \frac{1}{z}\right) + 16 = 0.$$

* Let $z = \cos\theta + i\sin\theta$.

From part (a): $10\cos 2\theta - 22\cos\theta + 16 = 0$

$$\Rightarrow 10(2\cos^2\theta - 1) - 22\cos\theta + 16 = 0$$

$$\Rightarrow 20\cos^2\theta - 22\cos\theta + 6 = 0$$

$$\Rightarrow 10\cos^2\theta - 11\cos\theta + 3 = 0$$

$$\Rightarrow (5\cos\theta - 3)(2\cos\theta - 1) = 0$$

$$\Rightarrow \cos\theta = \frac{3}{5} \text{ or } \cos\theta = \frac{1}{2}. \quad (M2) (A2)$$

Therefore, $\sin\theta = \pm\frac{4}{5}$ or $\sin\theta = \pm\frac{\sqrt{3}}{2}$.

This gives: $z = \frac{1}{5}(3 \pm 4i)$, $\frac{1}{2}(1 \pm i\sqrt{3})$. (A2)

Alternative (from *):

$$w = z + \frac{1}{z} \Rightarrow z^2 + \frac{1}{z^2} = w^2 - 2$$

$$5w^2 - 11w + 6 = 0 = (5w - 6)(w - 1)$$

$$\therefore z + \frac{1}{z} = 1 \text{ or } \frac{6}{5}$$

$$z^2 - z + 1 = 0 \Rightarrow z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\text{or } 5z^2 - 6z + 5 = 0 \Rightarrow z = \frac{3}{5} \pm \frac{4}{5}i$$

Section B

5. (i) (a) $a * (x \otimes b) = c$

From the * table: $a * b = c \Rightarrow x \otimes b = b$.

From the \otimes table: $c \otimes b = b$.

Therefore, $x = c$.

(R3)

(b) $(a \otimes x) * b = d$

From the * table: $b * b = d \Rightarrow a \otimes x = b$.

From the \otimes table: no x can be found for which $a \otimes x = b$.

Therefore, no x exists.

(R3)

[Note: A student who consistently reads the tables in the form "top row element" * (or \otimes) "left column element" may be awarded full marks for the answers (a) $x = a$
 (b) $x = b$.]

(ii) (a) $a^2 b a^2 b$

$$= a^2(ba^2)b \text{ (associativity)}$$

$$= a^2(ab)b \text{ (given)}$$

$$= a^3b^2 \text{ (associativity)}$$

$$= ee \text{ (given)}$$

$$= e \text{ (identity)}$$

(R3)

$$\begin{aligned}
 (b) \quad & a^2ba \\
 &= a(ab)a \quad (\text{associativity}) \\
 &= a(ba^2)a \quad (\text{given}) \\
 &= ab(a^3) \quad (\text{associativity}) \\
 &= abe \quad (\text{given}) \\
 &= ab \quad (\text{identity})
 \end{aligned} \tag{R3}$$

(iii) Let $\phi: G \rightarrow H$ be the one-one, onto function such that

$$\phi \left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right) = a + ib. \tag{M1}$$

$$\text{Thus } \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \rightarrow a_1 + ib_1 \text{ and } \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} \rightarrow a_2 + ib_2. \tag{A1}$$

$$\text{Now, } \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2 & -[a_1b_2 + a_2b_1] \\ a_1b_2 + a_2b_1 & a_1a_2 - b_1b_2 \end{pmatrix}, \tag{A1}$$

$$\text{and } (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1). \tag{A1}$$

$$\text{Therefore, } \phi \left(\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} \right) = (a_1 + ib_1)(a_2 + ib_2). \tag{A1}$$

Thus, G and H are isomorphic. (R1)

(iv) (a) Z_4 has 4 members $\{0, 1, 2, 3\}$ and Z_5 has 5 members $\{0, 1, 2, 3, 4\}$. *(M1)*

Therefore, $Z_4 \times Z_5$ has $4 \times 5 = 20$ members. *(M1) (A1)*

$$(3, 2) * (1, 4) = (3 + 1 \pmod{4}, 2 + 4 \pmod{5}) = (0, 1). \quad \text{*(M1) (A1)*}$$

(b) $Z_2 \times Z_3 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\} \quad \text{*(M1)*}$

Now, $(1, 1)^2 = (0, 2)$; $(1, 1)^3 = (1, 0)$; $(1, 1)^4 = (0, 1)$;
 $(1, 1)^5 = (1, 2)$; $(1, 1)^6 = (0, 0)$ which is the identity.

Therefore, $Z_2 \times Z_3$ is cyclic. *(R2)*

Both $(1, 1)$ and $(1, 2)$ are generators. *(A2)*

$$[(0, 1)^3 = (0, 0); (0, 2)^3 = (0, 0); (1, 2)^2 = (0, 0)] \quad \text{*(R1)*}$$

(c) $Z_2 \times Z_4 = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3)\} \quad \text{*(M1)*}$

$$(0, 1)^2 = (0, 2); (0, 1)^3 = (0, 3); (0, 1)^4 = (0, 0)$$

$$(0, 3)^2 = (0, 2); (0, 3)^3 = (0, 1); (0, 3)^4 = (0, 0)$$

$$(1, 3)^2 = (0, 2); (1, 3)^3 = (1, 1); (1, 3)^4 = (0, 0)$$

$$(1, 1)^2 = (0, 2); (1, 1)^3 = (1, 3); (1, 1)^4 = (0, 0)$$

$$(0, 2)^2 = (0, 0). \quad (1, 2)^2 = (0, 0). \quad (1, 0)^2 = (0, 0)$$

Therefore, $(0, 1)$, $(0, 3)$, $(1, 1)$ and $(1, 3)$ all have order 4. *(R4) (A2)*

6. (i) (a) $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}; B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$

(A2) (-1 per error to a max. of two).

(A2) (-1 per error to a max. of two).

(b) $A^2 = \begin{pmatrix} 3 & 2 & 3 & 3 & 3 \\ 2 & 4 & 4 & 2 & 4 \\ 3 & 4 & 5 & 3 & 5 \\ 3 & 2 & 3 & 3 & 3 \\ 3 & 4 & 5 & 3 & 5 \end{pmatrix}$

(A2) (-1 per error to a max. of two).

An entry a_{ij} in A^2 is the number of different walks between the vertices v_i and v_j which passes through exactly one vertex v_k which is the same as the number of paths of length 2 between v_i and v_j . (A1)

- (c) The sum of the entries in the j th column of A gives the **degree of the vertex v_j** provided there is no loop at that vertex. If there is one or more loops at vertex v_j , then $\deg(v_j) = j$ th col. sum - 1 + 2(no. of loops at v_j), since each loop contributes 2 to the degree of the vertex. (R1) (A1)

Note: Also accept $\deg(v_j) = j$ th col.sum + (no. of loops at v_j).

- (d) The column sum is 2 for each edge, since it joins exactly two vertices. The column sum is 1 for a loop. (R1) (A1)

- (ii) (a) In any loop-free, undirected graph, the maximum number of edges is $\binom{v}{2}$. (A1)

Hence, $e \leq \binom{v}{2} = \frac{v(v-1)}{2} \Rightarrow 2e \leq v^2 - v$. (A3)AG

- (b) In the loop-free, directed case, $e \leq v(v-1) = v^2 - v$. (A2)

- (iii) (a) Consider one vertex. There are $(n - 1)$ possible edges to start the path. Once we are at the second vertex, we have $(n - 2)$ possible edges; at the third vertex, $(n - 3)$ edges; at the fourth vertex, $(n - 4)$ edges. This gives $(n - 1)(n - 2)(n - 3)(n - 4)$ paths. Now, we have n vertices to choose as our starting vertex, and, according to the hint, we have counted each possible path twice.

$$\text{The required no. of paths} = \frac{n(n - 1)(n - 2)(n - 3)(n - 4)}{2}. \quad (R1) (A1)$$

- (b) Let e_1 be the number of edges in G , a simple undirected graph, and let e_2 be the number of edges in \bar{G} .

$$\text{Then } e_1 + e_2 = \binom{n}{2} = \frac{n(n - 1)}{2}, \text{ the number of edges in } k_n.$$

Since G is self-complementary, $e_1 = e_2$, so

$$e_1 = \frac{1}{2}\binom{n}{2} = \frac{n(n - 1)}{4}. \quad (R2) (A2)$$

- (c) The required example is:



The isomorphism from G to \bar{G} is as follows:

element	a	b	c	d
isomorphic image	b	d	a	c

(A2)

(d) Since G is self-complementary, it has $\frac{1}{2} \binom{n}{2} = \frac{n^2 - n}{4}$ edges.

Therefore, $4 \mid n(n - 1)$.

Now, if n is even, $n - 1$ is odd and so $4 \nmid n$.

i.e. $n = 4k, k \in N^*$.

(R2)AG

If $(n - 1)$ is even, n is odd, and so $4 \mid (n - 1)$.

i.e. $n - 1 = 4k$ or $n = 4k + 1, k \in N^*$.

(R2)AG

(iv) Without loss of generality, let the vertices and their degrees be:

vertex	a	b	c	d	e	f	g	h
degree	1	1	1	2	3	4	5	7

Vertex h has degree 7 and so must be connected to each of the other 7 vertices. Thus, a, b, c cannot be connected to any other vertex other than h . Vertex g has degree 5. It is connected already to h , and so must be connected to four other vertices. But there are only 3 (d, e, f) available. Therefore, no graph can be drawn.

(R3) (A2)

(v)

counter	vertex used	vertex added	weight added	cumulative
1		e	0	0
2	e	h	1	1
3	e, h	d	2	3
4	d, e, h	f	2	5
5	d, e, f, h	b	3	8
6	b, d, e, f, h	a	2	10
7	a, b, d, e, f, h	c	2	12
8	a, b, c, d, e, f, h	g	3	15
9	a, b, c, d, e, f, g, h	i	3	18
10	$a, b, c, d, e, f, g, h, i$		0	18

Therefore, the minimum cost = \$18 m.

(M5) (A1)

7. (i) The Poisson distribution with mean μ is defined by $p(x) = \frac{\mu^x e^{-\mu}}{x!}$
where x represents the number of arrivals during a given time interval.

$$(a) \text{ The required probability } = p(0) = \frac{3^0 e^{-3}}{0!} = e^{-3} = 0.0498. \quad (M3) (AI)$$

$$(b) \text{ The required probability} \\ = p(1) + p(2) = \frac{3e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!} = 0.373. \quad (M3) (AI)$$

- (ii) A t -distribution with $\bar{x} = 83.7$, $s = 12.9$, $n = 9$ and 8 degrees of freedom is appropriate here. $(RI) (AI)$

Since the standard deviation of the population is unknown, the standard deviation of \bar{x} is estimated with $\frac{s}{\sqrt{n}} = \frac{12.9}{3} = 4.3$. $(RI) (AI)$

The critical value associated with a 95% confidence interval in this two-tailed test is $t = 2.306$. $(A2)$

$$\text{Hence, the required interval} = \left(\bar{x} - t \frac{s}{\sqrt{n}}, \bar{x} + t \frac{s}{\sqrt{n}} \right) = (73.8, 93.6). \quad (M2) (A2)$$

Note: Some candidates will use instead of $\frac{s}{\sqrt{n}}$, $\frac{s}{\sqrt{n-1}}$, by using the unbiased estimator of population variance, viz. $s^2 \left(\frac{n}{n-1} \right)$ in place of s^2 . Then they will get the following:

Standard deviation of \bar{x} is estimated with

$$\frac{s}{\sqrt{n-1}} = \frac{12.9}{\sqrt{8}} = 4.56 \quad (RI) (AI)$$

The critical value of t here is 2.306. $(A2)$

$$\begin{aligned} \text{Hence, the required interval is} & \left(\bar{x} - t \frac{s}{\sqrt{n-1}}, \bar{x} + t \frac{s}{\sqrt{n-1}} \right) \\ & = (73.2, 94.2) \quad (M2) (A2) \end{aligned}$$

(iii) $H_0 : p \geq 0.4$

(A1)

$H_1 : p < 0.4$

The sample proportion is $p = \frac{25}{100} = 0.25$ with mean $\mu = 0.4$ and

(A1)

standard deviation $\sigma = \sqrt{\frac{(0.4)(0.6)}{100}} = 0.049.$

(A2)

If we assume a Normal distribution, then the corresponding z -value is

(R1) (A2)

$z = \frac{0.25 - 0.4}{0.049} = -3.062.$

Now, $p(Z < -3.062) = 0.0011$ which is less than 0.01.

(A2)

Therefore, the claim is rejected.

(A1)

- (iv) Let μ_A be the population mean for tyre *A* and μ_B that for tyre *B*. Since the sample size = 10, the *t*-distribution is appropriate.

The number of degrees of freedom = 9.

(R2)

- (a) For a 95% confidence interval, the appropriate value of $t = 2.262$.

(A2)

Hence, $\mu_A - \mu_B = 2000 \pm 2.262 \left(\frac{2000}{\sqrt{10}} \right) = 2000 \pm 1430.6$
miles.

(R1) (A3)

Therefore, $569 \leq \mu_A - \mu_B \leq 3430$ miles

(A2)

Note: Some candidates will use the unbiased estimator of population variance viz. $s^2 \left(\frac{n}{n-1} \right)$ to estimate the variance of \bar{x} . Then they will have

(R1) (A3)

$\mu_A - \mu_B = 2000 + 2.262 \left(\frac{2000}{\sqrt{9}} \right) = 2000 \pm 1508$

$\therefore 492 \leq \mu_A - \mu_B \leq 3508$ miles.

(A2)

- (b) We can accept H_0 if $\mu_A - \mu_B = 0$ falls in the 95% confidence interval. But this is clearly not so. Therefore, H_0 is rejected.

(R1) (A1)

8. (i) (a) If $e^x = x^4$, then $x = 4 \ln x$.

$$\text{Let } f(x) = 4 \ln x, \text{ then } f'(x) = \frac{4}{x}. \quad (A1)$$

Thus, $|f'(8)| = 0.5 < 1$, and so the given formula may be used.

$$\text{If } e^x = x^4, \text{ then } x = e^{x/4}. \quad (R1) (M1)$$

$$\text{Let } f(x) = e^{x/4}, \text{ then } f'(x) = \frac{1}{4} e^{x/4}. \quad (A1)$$

$$\text{Thus, } |f'(1)| = \frac{1}{4} e^{1/4} \approx 0.32 < 1, \text{ and so the formula may be used.} \quad (R1) (M1)$$

i	x_i	i	x_i
0	8	0	1
1	8.3178	1	1.2840
2	8.4736	2	1.3785
3	8.5478	3	1.4115
4	8.5827	4	1.4231
5	8.5990	5	1.4273
6	8.6066	6	1.4288
7	8.6101	7	1.4293
8	8.6118	8	1.4295
9	8.6125	9	1.4296
10	8.6129	10	1.4296
11	8.6130		
12	8.6131		
13	8.6131		

The required roots are 8.613 and 1.430 (3 dec pl.).

(A2)

(b) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ where $f(x) = e^x - x^4$, and $f'(x) = e^x - 4x^3$. (M1) (A1)

Thus, $x_1 = -1 - \left(\frac{e^{-1} - 1}{e^{-1} + 4} \right) = -1 - \left(\frac{1 - e}{1 + 4e} \right) = \frac{-2 - 3e}{1 + 4e}$. (A2)

(c) The area $= \int_0^1 (e^x - x^4) dx = \left[e^x - \frac{1}{5}x^5 \right]_0^1 = e - 1.2$. (M1) (A2)

(d) The area $\approx \frac{h}{3}(f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1))$,
 $h = 0.25$.

Thus, the area = 1.5178 (4 dec. pl.) (A1)

(e) The error term is $- \frac{(b-a)^5 f^{(4)}(c)}{180n^4}$ where $c \in [a, b]$. (A3)

The error term is $- \frac{1^5(e^c - 24)}{(180)(4^4)} < \frac{23}{(180)(256)} < 0.0005$, since
the minimum value of e^c is $e^0 = 1$. (R1) (A1)

Therefore, the maximum error = 0.0005. (A1)

Since estimated value + maximum error = 1.5178 + 0.0005 = 1.5183, and $e - 1.2 = 1.5183$ (4 dec. pl.), the answers to parts (c) and (d) are consistent with the calculated maximum error. (A1)

(f) We require the value of n for which $\frac{23}{180n^4} < 5 \times 10^{-6}$. (A1)

Thus, $n^4 > \frac{23}{900} \times 10^6 \Rightarrow n > 12.6$. (A2)

Therefore, the required value of n is 14. (n must be even) (R1) (A1)

(ii) (a) Let $f(x) = \frac{1}{x(\ln x)^2}$, $x \geq 2$. It is obvious that $f(x)$ is continuous on $[2, \infty)$ since $x(\ln x)^2$ is continuous and non-zero on $[2, \infty)$. (RI)

Clearly, $f(x) > 0$ for $x \in [2, \infty)$. (AI)

$$\text{Now, } f'(x) = -\frac{2x(\ln x)\frac{1}{x} + (\ln x)^2}{(x(\ln x)^2)^2} = -\frac{2 + \ln x}{x^2(\ln x)^3} < 0 \text{ for } x \geq 2. \quad (AI)$$

Therefore, $f(x)$ is decreasing on $[2, \infty)$ and so $f(x)$ satisfies the conditions under which the Integral Test may be used. (AI)

$$\begin{aligned} \text{Now, } \int_2^\infty \frac{dx}{x(\ln x)^2} &= \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x(\ln x)^2} \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{\ln x} \right]_2^b \\ &= \lim_{b \rightarrow \infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln b} \right) \\ &= \frac{1}{\ln 2} < \infty. \end{aligned}$$

Therefore, $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converges. (M1) (AI)

(b) Let $a_n = \frac{(-1)^n}{n}$. Then the series $\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \frac{1}{n}$ is divergent. (AI)

Also, the sequence $\{|a_n|\}$ is decreasing since

$$\frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)} < 0, \text{ and } \lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0. \quad (RI)$$

Thus, $\sum_{n=1}^\infty \frac{(-1)^n}{n}$ converges. (AI)

Therefore, $\sum_{n=1}^\infty \frac{(-1)^n}{n}$ converges conditionally. (AI)AG